Optimal matching, optimal transportation, and its regularity theory

The optimal matching of blue and red points is prima facie a combinatorial problem. It turns out that when the position of the points is random, namely distributed according to two independent Poisson point processes in d-dimensional space, the problem depends crucially on dimension, with the two-dimensional case being critical (Ajtai-Komlós-Tusnády).

Optimal matching is a discrete version of optimal transportation between the two empirical measures. While the matching problem was first formulated in its Monge version (p=1), the Wasserstein version (p=2) connects to a powerful continuum theory. This connection to a partial differential equation, the Monge-Ampère equation as the Euler-Lagrange equation of optimal transportation, enabled Parisi et. al. to give a finer characterization, made rigorous by Ambrosio et. al..

The idea of Parisi et. al. was to (formally) linearize the Monge-Ampère equation by the Poisson equation. I present an approach that quantifies this linearization on the level of the optimization problem, locally approximating the Wasserstein distance by an electrostatic energy. This approach (initiated with M. Goldman) amounts to the approximation of the optimal displacement by a harmonic gradient. Incidentally, such a harmonic approximation is analogous to de Giorgi's approach to the regularity theory for minimal surfaces. Because this regularity theory is robust — measures don't need to have Lebesgue densities — it allows for sharper statements on the matching problem (work with M. Huesmann and F. Mattesini).