Dealing with nonlocalities in variational functionals: Convexity notions, lower semicontinuity, and relaxation

Nonlocal variational problems arise in various applications, such as continuum mechanics, the theory of phase transitions, or image processing. Naturally, the presence of nonlocalities leads to new effects, and the standard methods in the calculus of variations, which tend to rely intrinsically on localization arguments, do not apply. In this talk, we address questions arising from the existence theory for three different classes of variational functionals: integrals depending on Riesz fractional gradients, double integrals, and double supremals - and find qualitatively very different results. Regarding the characterization of weak lower semicontinuity, it may be surprising that quasiconvexity, which is well-known from the classical local setting, also provides the correct convexity notion for the fractional integrals. Our proof relies on a translation mechanism that allows switching between classical and fractional gradients. In the case of double supremals, we show that the natural guess of separate level convexity fails in the vectorial case, and introduce the new Cartesian level convexity. As for relaxation, we discuss the central issue of why one cannot expect these nonlocal functionals, in contrast to their local counterparts, to be structure-preserving. This is based on joint work with Antonella Ritorto, Hidde Schönberger (both KU Eichstätt-Ingolstadt), and Elvira Zappale (Sapienza University of Rome).